Patrolling Repairman Example

- N machines
- Single repairman visits machines in order $1 \to 2 \to \cdots \to N \to 1 \to 2 \to \cdots$
- Repairs stopped machine, walks past running machine
- Repair times for machine j are i.i.d. as a random variable R_j
- Lifetimes for machine j are i.i.d. as a continuous random variable L_j
- Walking time from machine j to next machine is a constant $W_j > 0$
- At time 0, the repairman has just finished repairing machine 1 and all other machines are broken.

Suppose we wish to estimate μ_r, the expected fraction of time in [0, t] that the repairman spends repairing machines. If we define our system state by X(t) = A(t), where

$$A(t) = \begin{cases} 1 & \text{if repairman is repairing a machine} \\ 0 & \text{otherwise} \end{cases}$$

then $\mu_r = E\left[\frac{1}{t}\int_0^t A(u)\,du\right]$. We might also want to estimate μ_s, the expected number of stopped machines at time t, or μ_w, the long-run average wait for repair for machine 1.

Problems:

- Can't determine number of stopped machines just from observing A(t)
- Not even clear how to generate sample paths of $\{A(t): t \ge 0\}$

$\Rightarrow$ need to put more information into state definition

Here's another attempt at a state definition:

$$X(t) = (Z_1(t), Z_2(t), \ldots, Z_N(t), M(t), N(t)),$$

where

$$Z_j(t) = \begin{cases} 1 & \text{if machine j is waiting for repair at time t} \\ 0 & \text{otherwise} \end{cases}$$

$$M(t) = \begin{cases} j & \text{if machine j is under repair at time t} \\ 0 & \text{if no machine is under repair at time t} \end{cases}$$

N(t) = j if at time t the repairman will next arrive at machine j

Then we can generate sample paths of $\{X(t)\colon t \geq 0\}$ (because this process is a well-defined GSMP as shown below and, as discussed in class, there is a well-defined algorithm for generating sample paths of a GSMP). Also, all of the system characteristics of interest can be precisely expressed in terms of $\{X(t)\colon t \geq 0\}$:

$$\mu_r = E\left[\frac{1}{t}\int_0^t f_r(X(u))\,du\right] \quad \text{and} \quad \mu_s = E[f_s(X(t))]$$

where

$$f_r(z_1, \ldots, z_N, m, n) = 1_{\{1,2,\ldots,N\}}(m)$$

$$f_s(z_1, \ldots, z_N, m, n) = z_1 + z_2 + \ldots + z_N + 1_{\{1,2,\ldots,N\}}(m)$$

(Here $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise.)

Also, we can express μ_w in terms of $\{X(t)\colon t \geq 0\}$. To see this, set $B_0 = 0$ and recursively define the start and termination of the nth waiting time for machine 1 by

$$A_n = \min\{\zeta_k > B_{n-1}\colon Z_1(\zeta_{k-1}) = 0 \text{ and } Z_1(\zeta_k) = 1\} \text{ and } B_n = \min\{\zeta_k > A_n\colon M(\zeta_{k-1}) \neq 1 \text{ and } M(\zeta_k) = 1\}$$

where ζ_n is the time of the n^{th} state transition. We can also define these times in terms of the continuous time process by setting

$$A_n = \min\{t > B_{n-1}\colon Z_1(t-) = 0 \text{ and } Z_1(t) = 1\} \text{ and } B_n = \min\{t > A_n\colon M(t-) \neq 1 \text{ and } M(t) = 1\},$$

where X(t-) indicates the state of the system just before time t.

In either case, we can then write the n^{th} waiting time as $D_n = B_n - A_n$, and hence

$$\mu_w = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} D_k \quad \text{(assuming that it exists)}$$

The process $\{X(t)\colon t \geq 0\}$ can be specified as a GSMP as follows:

- S consists of all $(z_1, \ldots, z_N, m, n) \in \{0,1\}^N \times \{0, 1, \ldots, N\} \times \{1, 2, \ldots, N\}$ such that
  - $n = m + 1$ if $0 < m < N$
  - $n = 1$ if $m = N$
  - $m = j$ only if $z_j = 0$ $(1 \leq j \leq N)$
- $E = \{e_1, e_2, \ldots, e_{N+2}\}$, where
  - e_j = "stoppage of machine j" $(1 \leq j \leq N)$
  - e_{N+1} = "completion of repair"
  - e_{N+2} = "arrival of repairman"
- E(s) is defined as follows for $s = (z_1, \ldots, z_N, m, n)$:

- ♦ $e_j \in E(s)$ $(1 \le j \le N)$ iff $z_j = 0$ and $m \ne j$
- ♦ $e_{N+1} \in E(s)$ iff $m > 0$
- ♦ $e_{N+2} \in E(s)$ iff $m = 0$
- $p(s'; s, e^*)$ is defined as follows:
  - ♦ if $e^* = e_j$ $(1 \le j \le N)$, then $p(s'; s, e^*) = 1$
    when $s = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_N, m, n)$ and $s' = (z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_N, m, n)$
  - ♦ if $e^* = e_{N+2}$, then $p(s'; s, e^*) = 1$
    when $s = (z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_N, 0, j)$ with $j < N$ and $s' = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_N, j, j+1)$;
    when $s = (z_1, z_2, \ldots, z_{N-1}, 1, 0, N)$ and $s' = (z_1, z_2, \ldots, z_{N-1}, 0, N, 1)$;
    when $s = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_N, 0, j)$ with $j < N$ and $s' = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_N, 0, j+1)$;
    and when $s = (z_1, z_2, \ldots, z_{N-1}, 0, 0, N)$ and $s' = (z_1, z_2, \ldots, z_{N-1}, 0, 0, 1)$
  - ♦ exercise: do the case $e^* = e_{N+1}$
- $F(x; s', e', s, e^*)$ is defined as follows
  - ♦ if $e' = e_j$ $(1 \le j \le N)$, then $F(x; s', e', s, e^*) = P\{L_j \le x\}$
  - ♦ if $e' = e_{N+1}$ and $s' = (z_1, \ldots, z_N, m, n)$ then $F(x; s', e', s, e^*) = P\{R_m \le x\}$
  - ♦ if $e' = e_{N+2}$ and $s' = (z_1, \ldots, z_N, 0, n)$ then $F(x; s', e', s, e^*) = 1_{[0,x]}(W_{n-1})$ if $n > 1$ and $1_{[0,x]}(W_N)$ if $n = 1$
- $r(s, e) \equiv 1$ for all s and e
- initial dist'n: $\nu(s) = 1$, where $s = (0,1,1,\ldots,1,0,2)$, $F_0(x; e_1, s) = P\{L_1 \le x\}$ and $F_0(x; e_{N+2}, s) = 1_{[0,x]}(W_1)$